Global Orbit Patterns for Dynamical Systems On Finite Sets

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Abstract. In this paper, the study of the global orbit pattern (gop) formed by all the periodic orbits of discrete dynamical systems on a finite set X allows us to describe precisely the behaviour of such systems. We can predict by means of closed formulas, the number of gop of the set of all the function from X to itself. We also explore, using the brute force of computers, some subsets of locally rigid functions on X, for which interesting patterns of periodic orbits are found.

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INTRODUCTION

In some engineering applications such as chaotic encryption, chaotic maps have to exhibit required statistical and spectral properties close to those of random signals. There is a growing industrial interest to consider and study thoroughly the property of such map [10, 11, 12].

Very often, dynamical systems in several dimensions are obtained coupling 1-dimension ones and their properties are strongly linked [5].

Quasi-periodic or chaotic motion is frequently present in complicated dynamical systems whereas simple dynamical systems often involve only periodic motion. The most famous theorem in this field of research is the Sharkovskii's theorem, which addresses the existence of periodic orbits of continuous maps of the real line into itself. This theorem was once proved toward the year 1962 and published only two years after [4].

Mathematical results concerning periodic orbits are often obtained for functions on real intervals. However, most of the time, as the complex behaviour of chaotic dynamical systems is not explicitly tractable, mathematicians have recourse to computer simulations. The main question which arises then is: does these numerical computations are reliable ?

As an example we report the results of some computer experiments on the orbit structure of the discrete maps on a finite set which arise when the logistic map is iterated "naively" on the computer.

Due to the discrete nature of floating points used by computers, there is a huge gap between these results and the theoretical results obtained when this map is considered on a real interval. This gap can be narrowed in some sense (i.e. avoiding the collapse of periodic orbits) in higher dimensions when ultra weak coupling is used [6, 7].

Nowadays the claim is to understand precisely which periodic orbit can be observed numerically in such systems. In a first attempt we study in this paper the orbits generated by the iterations of a one-dimensional system on a finite set X_N with a cardinal N. The final goal of a good understanding of the actual behaviour of dynamical systems acting on floating numbers (i.e. the numbers used by computers) will be only reached after this first step will be achieved.

On finite set, only periodic orbits can exist. For a given function we can compute all the orbits, all together they form a global orbit pattern. We formalise such a gop as the ordered set of periods when the initial value thumbs the finite set in the increasing order. We are able to predict, using closed formulas, the number of gop for the set $\mathscr{F}_N$ of all the functions on X. We also explore by computer experiments special subsets of $\mathscr{F}_N$, such as sets of locally "rigid" functions which presents interesting patterns of gop.

This article is organized as follows : in the section "Computational divergences" we display some examples of such computational divergences for the logistic map in various ways of discretization. In the section "Pattern defined by all the orbits of a dynamical system" we introduce a new mathematical tool: the global orbit pattern, in order to describe more precisely the behaviour of dynamical systems on finite sets. In the section "Cardinal of classes" we give some closed formulas related to the cardinal of classes of gop of $\mathscr{F}_N$. In the section "Functions with local properties" we study the case of sets of functions with a kind of local "rigidity" versus their gop, in order to show the usefulness of these new tools.

COMPUTATIONAL DIVERGENCES

Discretized logistic map

As an example of collapsing effects which happen when using computers in numerical experiments, we presents the results of a sampling study in double precision of a discretization of the logistic map $f_4 : [0,1] \to [0,1]$ (see Fig. 1)

$$f_4(x) = 4x(1-x) \tag{1}$$

and its associated dynamical system

$$x_{n+1} = 4x_n(1-x_n) \tag{2}$$

which has excellent ergodic properties on the real interval.

There exists an unstable fixed point 0.

The set $\left\{\frac{5-\sqrt{5}}{8}, \frac{5+\sqrt{5}}{8}\right\} = \{0.3454915, 0.9045084\}$ is the period-2 orbit.

In fact there exist an infinity of periodic orbits and an infinity of periods. This dynamical system possesses an invariant measure (see Fig. 2):

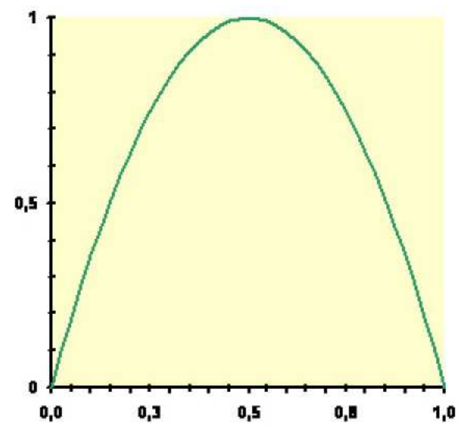

FIGURE 1. Graph of the map $f(x) = 4x(1-x)$ on $[0,1]$

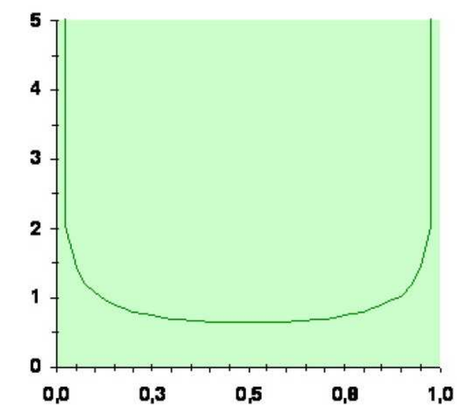

FIGURE 2. Invariant measure of the logistic map

$$P(x) = \frac{1}{\pi\sqrt{x(1-x)}} \tag{3}$$

However, in numerical computations using ordinary (IEEE-754) double precision numbers - so that the working interval contains of the order of 10^{16} representable points - out of 1,000 randomly chosen initial points (see Table 1),

- 596, i.e., the majority, converged to the fixed point corresponding to the unstable fixed point $\{0\}$ in equation 2,
- 404 converged to a cycle of period 15,784,521.

Thus, in this case at least, the very long-term behaviour of numerical orbits is, for a substantial fraction of initial points, in flagrant disagreement with the true behaviour of typical orbits of the original smooth logistic map.

In others numerical experiments we have performed, the computer working with fixed finite precision is able to represent finitely many points in the interval in question. It is probably good, for purposes of orientation, to think of the case where the representable points are uniformly spaced in the interval. The true logistic map is then *approximated* by a discretized map, sending the finite set of representable points in the interval to itself.

Describing the discretized mapping exactly is usually complicated, but it is *roughly* the mapping obtained by applying the exact smooth mapping to each of the discrete representable points and "rounding" the result to the nearest representable point. In our experiments uniformly spaced points in the interval with several order of discretization (ranging from 9 to 2,001 points) are involved. In each experiment the questions addressed are:

- how many periodic cycles are there and what are their periods ?
- how large are their respective basins of attraction, i.e. , for each periodic cycle, how many initial points give orbits with eventually land on the cycle in question ?

TABLE 1. Coexisting periodic orbits found using 1,000 random initial points for double precision numbers

Period	Orbit	Relative Basin size
1	$\{0\}$ (unstable fixed point)	596 over 1,000
15,784,521	Scattered over the interval	404 over 1,000

TABLE 2. Coexisting periodic orbits for the discretization with regular meshes of $N = 9$, 10 and 11 points

N	Period	Orbit	Basin size
9	1	$\{0\}$	3 over 9
9	1	$\{6\}$	2 over 9
9	1	$\{3,7\}$	4 over 9
10	1	$\{0\}$	2 over 10
10	2	$\{3,8\}$	8 over 10
11	1	$\{0\}$	3 over 11
11	4	$\{3,8,6,9\}$	8 over 11

TABLE 3. Coexisting periodic orbits for the discretization with regular meshes of $N = 99$, 100 and 101 points

N	Period	Orbit	Basin size
99	1	$\{0\}$	3 over 99
99	10	$\{3,11,39,93,18,58,94,15,50,97\}$	96 over 99
100	1	$\{0\}$	2 over 100
100	1	$\{74\}$	2 over 100
100	6	$\{11,39,94,18,58,96\}$	72 over 100
100	7	$\{7,26,76,70,82,56,97\}$	24 over 100
101	1	$\{0\}$	3 over 101
101	1	$\{75\}$	2 over 101
101	1	$\{16,61,95\}$	96 over 101

On an another hand, for relatively coarse discretizations the orbit structure is determined completely, i.e., all the periodic cycles and the exact sizes of their basins of attraction are found. Some representative results are given in Tables 2 to 4. In theses tables, N

TABLE 4. Coexisting periodic orbits for the discretization with regular meshes of $N = 1,999;2,000$ and $2,001$ points

N	Period	Orbit	Basin size
1,999	1	$\{0\}$	3 over 1,999
1,999	4	$\{554; 1,601; 1,272; 1,848\}$	990 over 1,999
1,999	8	$\{3; 11; 43; 168; 615; 1,702; 1,008; 1,997\}$	1,006 over 1,999
2,000	1	$\{0\}$	2 over 2,000
2,000	1	$\{1,499\}$	14 over 2,000
2,000	2	$\{691; 1,808\}$	138 over 2,000
2,000	3	$\{276; 1,221; 1,900\}$	6 over 2,000
2,000	8	$\{3; 11; 43; 168; 615; 1,703; 1,008; 1,998\}$	1,840 over 2,000
2,001	1	$\{0\}$	5 over 2,001
2,001	1	$\{1,500\}$	34 over 2,001
2,001	2	$\{691; 1,809\}$	92 over 2,001
2,001	8	$\{3; 11; 43; 168; 615; 1,703; 1,011; 1,999\}$	608 over 2,001
2,001	18	$\{35; 137; 510; 1,519; 1,461; 1,574; \ldots\}$	263 over 2,001
2,001	25	$\{27; 106; 401; 1,282; 1,840; 588; \ldots\}$	1,262 over 2,001

denotes the order of the discretization, i.e., the representable points are the numbers, $\frac{j}{N}$, with $0 \leq j < N$.

The Table 2 shows coexisting periodic orbits for the discretization with regular meshes of $N = 9$, 10 and 11 points. There are exactly 3, 2 and 2 cycles.
The Table 3 shows coexisting periodic orbits for the discretization with regular meshes of $N = 99$, 100 and 101 points. There are exactly 2, 4 and 3 cycles.
The Table 4 shows coexisting periodic orbits for the discretization with regular meshes of $N = 1,999$, $N = 2,000$ and $N = 2,001$ points.

It seems that the computation of numerical approximations of the periodic orbits leads to unpredictable results.

Statistical properties

Many others examples could be given, but those given may serve to illustrate the intriguing character of the results: the outcomes proves to be extremely sensitive to the details of the experiment, but the results all have a similar flavour : a relatively small number of cycles attract near all orbits, and the lengths of these significant cycles are much larger than one but much smaller than the number of representable points.

In [1], P. Diamond and A. Pokrovskii, suggest that statistical properties of the phenomenon of computational collapse of discretized chaotic mapping can be modelled by random mappings with an absorbing centre. The model gives results which are very much in line with computational experiments and there appears to be a type of universality summarised by an Arcsine law. The effects are discussed with special reference to the family of mappings

$$x_{n+1} = 1 - |1 - 2x_n|^{\ell} \quad 0 \leq x \leq 1 \quad 1 \leq \ell \leq 2 \tag{4}$$

Computer experiments show close agreement with prediction of the model.

However these results are of statistical nature, they do not give accurate information on the exact nature of the orbits (e.g. length of the shortest one, of the greater one, size of their basin of attraction ...). It is why we consider the problem of computational discrepancies in an original way in the next section.

PATTERN DEFINED BY ALL THE ORBITS OF A DYNAMICAL SYSTEM

In this section in order to describe more precisely which kind of behaviour occurs in discretized dynamical systems on finite sets we conceive a new mathematical tool: the global orbit pattern of a function that is the set of the periods of every different orbits of the dynamical system associated to the function when the initial points are took in increasing order.

General definitions

For every $x_0 \in X$, let $\{x_i\}$ be the sequence of the orbit of the dynamical system associated to the function f which maps X onto X defined by

$$x_{i+1} = f(x_i) \;\text{ for }\; i \geq 0. \tag{5}$$

For convenience $\forall x_0 \in X$ we denote

$$f^0(x_0) = x_0 \tag{6}$$

and

$$\forall p \geq 1, \forall x_0 \in X, \;\; f^p(x_0) = \underbrace{f \circ f \circ \ldots \circ f}_{p \text{ times}}(x_0). \tag{7}$$

Hence

$$x_i = f^i(x_0). \tag{8}$$

The orbit of x_0 under f is the set of points $\mathscr{O}(x_0, f) = \{f^i(x_0), i \geq 0\} = \{x_i, i \geq 0\}$.
The starting point x_0 for the orbit is called the initial value of the orbit.
A point x is a fixed point of the map f if $f(x) = x$.
A point x is a periodic point with period p if $f^p(x) = x$ and $f^k(x) \neq x$ for all k such that $0 \leq k < p$, p is called the order of x.
If x is periodic of order p, then the orbit of x under f is the finite set $\{x, f(x), f^2(x), \ldots, f^{p-1}(x)\}$. We will call this set the periodic orbit of order p or a p-cycle.
A fixed point is then a 1-cycle.
The point x is an eventually periodic point of f with order p if there exists $K > 0$ such that $\forall k \geq K \; f^{k+p}(x) = f^k(x)$.
$\forall x \in X$, we denote $\omega(x, f)$ the order of x under f or simply $\omega(x)$ when the map f involved is obvious.

A subset T of X is invariant under f if $f^{-1}(T) = T$. That is equivalent to say that T is invariant under f if and only if $f(T) \subset T$ and $f^{-1}(T) \subset T$.

Notation $\sharp X$ is the cardinal of the finite set X.

Map on finite set

Along this paper, N is a non-zero integer and $\sharp A$ stands for the cardinal of any finite set A. In this article, we consider X as an ordered finite set with N elements. We denote it X_N, it is isomorphic to the interval $[\![0, N-1]\!] \subset \mathbb{N}$. Then $\sharp X_N = N$. Let f be a map from X_N into X_N. We denote by $\mathscr{F}_N$ the set of the maps from X_N into X_N. Clearly, $\mathscr{F}_N$ is a finite set and $\sharp \mathscr{F}_N = N^N$ elements. For all $x \in X_N$, $\mathscr{O}(x, f)$ is necessarily a finite set with at most N elements. Indeed, let us consider the sequence $\{x, f(x), f^2(x), \ldots, f^{N-1}(x), f^N(x)\}$ of the first $N+1$ iterated points. Thanks to the Dirichlet's box principle, two elements are equals because X_N has exactly N different values. Thus, every initial value of X_N leads ultimately to a repeating cycle. More precisely, if x is a fixed point $\mathscr{O}(x, f)$ is the unique element x and if x is a periodic point with order p, $\mathscr{O}(x, f)$ has exactly p elements. In this case, the orbit of x under f is the set $\mathscr{O}(x, f) = \{x, f(x), f^2(x), \ldots, f^{p-1}(x)\}$. If x is an eventually periodic point with order p, there exists $K > 0$ such that $\forall k \geq K \; f^{k+p}(x) = f^k(x)$. In this case, the orbit of x under f is the set $\mathscr{O}(x, f) = \{x, f(x), f^2(x), \ldots, f^K(x), f^{K+1}(x), \ldots, f^{K+p-1}(x)\}$.

Equivalence classes

Components

Let $f \in \mathscr{F}_N$. We consider on X_N the relation $\sim$ defined by : $\forall x, x' \in X_N, \;\; x \sim x' \Leftrightarrow \exists k \in \mathbb{N}$ such that $f^k(x) \in \mathscr{O}(x', f)$. The relation $\sim$ is an equivalence relation on X_N. $\mathscr{S}_N/\sim$ is the collection of the equivalence classes that we will call components of X_N under f which constitute a partition of X_N. The number of components are given in [3]. Asymptotic properties of the number of cycles and components are studied in [8]. For each component, we take as class representative element the least element of the component. The components will be written $T_N(x_0, f), \ldots, T_N(x_{p_{f,N}}, f)$ where x_i is the least element of $T_N(x_i, f)$.

By analogy with real dynamical systems, we can define attractive and repulsive components in discretized dynamical systems as follows.

***Definition* 1** A component is repulsive when it is a cycle. Otherwise, the component is attractive.

Remark In other words, a component is attractive when the component contains at least an eventually periodic element. The corresponding cycle is strictly contained in an attractive component.

Examples are given in Tables 5, 6 and 7.

For instance, in Table 6, the fonction f has $\{2,7\}$ as period-2 orbit and $\{1,2,7,9\}$ as component which is attractive because 1 and 2 are eventually periodic elements.

TABLE 5. Orbits and components of a function belonging to $\mathscr{F}_{11}$ with gop $[2,2,1,3]_{11}$.

Function			orbit/component/nature		
0	$\to$	6			
1	$\to$	3			
2	$\to$	2	period-2 orbit : $\{6,9\}$	$\{0,6,9\}$	attractive
3	$\to$	5			
4	$\to$	8	period-2 orbit : $\{5,10\}$	$\{1,3,5,10\}$	attractive
5	$\to$	10			
6	$\to$	9	fixed point : $\{2\}$	$\{2\}$	repulsive
7	$\to$	4			
8	$\to$	7	period-3 orbit : $\{4,8,7\}$	$\{4,8,7\}$	repulsive
9	$\to$	6			
10	$\to$	5			

TABLE 6. Orbits and components of a function belonging to $\mathscr{F}_{11}$ with gop $[2,2,1,3]_{11}$.

Function			orbit/component/nature		
0	$\to$	4			
1	$\to$	2			
2	$\to$	7	period-2 orbit : $\{4,8\}$	$\{0,4,8\}$	attractive
3	$\to$	3			
4	$\to$	8	period-2 orbit : $\{2,7\}$	$\{1,2,7,9\}$	attractive
5	$\to$	10			
6	$\to$	5	fixed point : $\{3\}$	$\{3\}$	repulsive
7	$\to$	2			
8	$\to$	4	period-3 orbit : $\{5,10,6\}$	$\{5,10,6\}$	repulsive
9	$\to$	1			
10	$\to$	6			

Order of elements

Here are some remarks on the order of elements of components.

Remark The order of every element of a component is the length of its inner cycle.

TABLE 7. Orbits and components of a function belonging to $\mathscr{F}_{11}$ with gop $[2,2,1,3]_{11}$.

Function			orbit/component/nature		
0	$\to$	9			
1	$\to$	6			
2	$\to$	4	period-2 orbit : $\{0,9\}$	$\{0,9\}$	repulsive
3	$\to$	7			
4	$\to$	10	period-2 orbit : $\{1,6\}$	$\{1,6\}$	repulsive
5	$\to$	3			
6	$\to$	1	fixed point : $\{10\}$	$\{2,4,8,10\}$	attractive
7	$\to$	5			
8	$\to$	2	period-3 orbit : $\{3,7,5\}$	$\{3,7,5\}$	repulsive
9	$\to$	0			
10	$\to$	10			

***Definition* 2** For all $x \in X_N$, there exists $i \in [\![0, p_{f,N}]\!]$ such that x belongs to the component $T_N(x_i, f)$. Then $\omega(x, f)$ is equal to the order $\omega(x_i, f)$.

Remark For all $i \in [\![0, p_{f,N}]\!]$, $T_N(x_i, f)$ is an invariant subset of X_N under f.

In the example given in Table 5, the order of the element 0 is 2, the order of the element 1 is 2, the order of the element 4 is 3. The elements 1 and 3 have the same order.

Definition of global orbit pattern

For each $f \in \mathscr{F}_N$, we can determine the components of X_N under f. For each component, we determine the order of any element. Thus, for each $f \in \mathscr{F}_N$, we have a set of orders that we will denote $\Omega(f,N)$. Be given f, there exist $p_{f,N}$ components and $p_{f,N}$ representative elements such that $x_0 < x_1 < \ldots < x_{p_{f,N}}$.

For each $f \in \mathscr{F}_N$, the sequence $[\omega(x_0), \omega(x_1), \ldots, \omega(x_{p_{f,N}}); f]_{\mathscr{F}_N}$ with $x_0 < x_1 < \ldots < x_{p_{f,N}}$ will design the global orbit pattern of $f \in \mathscr{F}_N$.

We will write $gop(f) = [\omega(x_0), \omega(x_1), \ldots, \omega(x_{p_{f,N}}); f]_{\mathscr{F}_N}$.

When the reference to $f \in \mathscr{F}_N$ is obvious, we will write shortly
$gop(f) = [\omega(x_0), \omega(x_1), \ldots, \omega(x_{p_{f,N}})]_N$ or $gop(f) = [\omega(x_0), \omega(x_1), \ldots, \omega(x_{p_{f,N}})]$.

For example, the same gop associated to the functions given in Tables 5, 6 and 7 is $[2,2,1,3]_{11}$.

Another example is given in Table 8. In that example, we have $\omega(0) = 2$, $\omega(3) = 1$, $\omega(4) = 4$.

TABLE 8. Orbits and components of a function belonging to $\mathscr{F}_8$ with gop $[2,1,4]_8$.

Function			orbit/component/nature		
0	$\rightarrow$	1			
1	$\rightarrow$	0	period-2 orbit : $\{0,1\}$	$\{0,1,2\}$	attractive
2	$\rightarrow$	0			
3	$\rightarrow$	3	fixed point : $\{3\}$	$\{3\}$	repulsive
4	$\rightarrow$	5			
5	$\rightarrow$	6	period-4 orbit : $\{4,5,6,7\}$	$\{4,5,6,7\}$	repulsive
6	$\rightarrow$	7			
7	$\rightarrow$	4			

***Definition* 3** The set of all the global orbit patterns of $\mathscr{F}_N$ is called $\mathscr{G}(\mathscr{F}_N)$.

For example, for $N=5$, the set $\mathscr{G}(\mathscr{F}_5)$ is
$\{[1];[1,1];[1,1,1];[1,2];[1,1,1,1];[1,1,2];[1,3];[1,1,1,1,1];[1,1,1,2];[1,1,2,1];$
$;[1,2,1,1];[1,2,2];[1,3,1];[1,4];;[2,1];[2,1,1];[2,2];[2,1,1,1];[2,1,2];[2,2,1]$
$;[3];[3,1];[3,1,1];[3,2];[4];[4,1];[5]\}$.

Class of gop

We give the following definitions :

***Definition* 4** Let be $A=[\omega_1,\ldots,\omega_p]_N$ a gop. Then the class of A, written $\overline{A}$, is the set of all the functions $f \in \mathscr{F}_N$ such that the global orbit pattern associated to f is A.

For example, for $N=11$, the class of the gop $\overline{[2,2,1,3]}_{11}$ contains the following few of many functions defined in Tables 5, 6 and 7. The periodic orbit which are encountered have the same length nevertheless there are different.

***Definition* 5** Let be $A=[\omega_1,\ldots,\omega_p]_N$ a gop.
Then the modulus of A is $|A| = \sum\limits_{k=1}^{p} \omega_k$.

Remark $\big|[\omega_1,\ldots,\omega_p]_N\big| \leq N$.

Notation $[\omega_{\tilde{k}}]_N$ means $[\underbrace{\omega,\ldots,\omega}_{k \text{ times}}]_N$ and $[\omega_{\tilde{k}},\nu_{\tilde{m}}]_N$ means $[\underbrace{\omega,\ldots,\omega}_{k \text{ times}},\underbrace{\nu,\ldots,\nu}_{m \text{ times}}]_N$.

Threshold functions

Ordering the discrete maps

***Theorem* 1** The sets $\mathscr{F}_N$ and $[\![1, N^N]\!]$ are isomorphic.

Proof We define the function ϕ from $\mathscr{F}_N$ to $[\![1, N^N]\!]$ by : for each $f \in \mathscr{F}_N$, $\phi(f)$ is the integer n such that $n = \sum_{k=0}^{N-1} f(k) N^{N-1-k} + 1$.
Then ϕ is well defined because $n \in [\![1, N^N]\!]$.
Let n be a given integer between 1 and N^N. We convert $n-1$ in base N : there exists a unique N-tuple $(a_{n-1,0}; a_{n-1,1}; \ldots; a_{n-1,N-1}) \in [\![0, N-1]\!]^N$ such that $\overline{n-1}^N = \sum_{i=0}^{N-1} a_{n-1,N-1-i} N^{N-i-1}$. We can thus define the map f_n with : $\forall i \in X_N$, $f_n(i) = a_{n-1,N-i-1}$. Then ϕ is one to one.

Remark This implies $\mathscr{F}_N$ is totally ordered.

***Definition* 6** Let $f \in \mathscr{F}_N$. Then

$$n = \sum_{k=0}^{N-1} f(k) N^{N-1-k} + 1 \tag{9}$$

is called the rank of f.

Threshold functions

Be given a global orbit pattern A, we are exploring the class $\overline{A}$.

***Theorem* 2** For every $A \in \mathscr{G}(\mathscr{F}_N)$, the class $\overline{A}$ has a unique function with minimal rank.

***Definition* 7** For every class $\overline{A} \in \mathscr{G}(\mathscr{F}_N)$, the function defined by the previous theorem will be called the threshold function for the class $\overline{A}$ and will be denoted by $Tr(A)$ or $Tr(\overline{A})$.

To prove the theorem, we need the following definition :

***Definition* 8** Let $f \in \mathscr{F}_N$ be a function. Let p a non zero integer smaller than N. Let be $x_1, \ldots, x_p$ p consecutive elements of X_N. Then $x_1, \ldots, x_p$ is a canonical p-cycle in relation to f if $\forall j \in [\![1, p-1]\!]$, $f(x_j) = x_{j+1}$ and $f(x_p) = x_1$.

Proof Let $[\omega_1,\ldots,\omega_p]$ be a global orbit pattern of $\mathscr{G}(\mathscr{F}_N)$. We construct a specific function f belonging to the class $\overline{[\omega_1,\ldots,\omega_p]}$ and we prove that the function so obtained is the smallest with respect to the order on $\mathscr{F}_N$. With the first ω_1 elements of $[\![0,N-1]\!]$, that is the set of integers $[\![0,\omega_1-1]\!]$, we construct the canonical ω_1-cycle : if $\omega_1=1$, we define $f(0)=0$, else $f(0)=1$, $f(1)=2$, ..., $f(\omega_1-2)=\omega_1-1$, $f(\omega_1-1)=0$.
Then $\forall j \in [\![\omega_1-1,\omega_1+N-s-1]\!]$, we define $f(j)=0$.
Then with the next ω_2 integers $[\![\omega_1+N-s,\omega_1+N-s+\omega_2-1]\!]$ we construct the canonical ω_2-cycle. We keep going on constructing for all $j\in[\![3,p]\!]$ the canonical ω_j-cycle.
In consequence, we have found a function f belonging to the class $\overline{[\omega_1,\ldots,\omega_p]}$.
Assume there exists a function $g\in\mathscr{F}_N$ belonging to the class of f such that $g<f$. Let $I=\{i\in[\![0,N-1]\!]$ such that $f(i)\neq 0\}$. As $g<f$, there exists $i_0\in I$ such that $g(i_0)<f(i_0)$. There exists also j_0 such that $i_0\in\omega_{j_0}$. If $f(i_0)=i_0$, then $\omega_{j_0}=1$, $g(i_0)<i_0$ and then $g(i_0)\notin\omega_{j_0}$. Then the global orbit pattern of g doesn't contain anymore 1 as cycle. The global orbit pattern of g is different from the global orbit pattern of f. If $f(i_0)=i_0+1$, then $g(i_0)\le i_0$. Either $g(i_0)=i_0$ and then the global orbit pattern of g is changed, or $g(i_0)<i_0$ and we are in the same situation as previously. Thus, in any case, the smallest function belonging to the class $\overline{[\omega_1,\ldots,\omega_p]}$ is the one constructed in the first part of the proof.

The proof of the theorem gives an algorithm of construction of the threshold function associated to a given gop.
The threshold function associated to the gop $[2_{\tilde{2}},1,3]_{11}$ is explained in Table 9. Its rank is $n=25,938,474,637$.

TABLE 9. Algorithm for the threshold function construction for the gop $[2_{\tilde{2}},1,3]_{11}$.

First step	Second step	Third step	Fourth step	Fifth step
Construction of the first canonical 2-cycle	Construction of the last canonical 3-cycle	Construction of the canonical 1-cycle	Construction of the canonical 2-cycle	Filling the remaining images with 0
$0 \to 1$	$0 \to 1$	$0 \to 1$	$0 \to 1$	$0 \to 1$
$1 \to 0$	$1 \to 0$	$1 \to 0$	$1 \to 0$	$1 \to 0$
$2 \to$	$2 \to$	$2 \to$	$2 \to$	$2 \to 0$
$3 \to$	$3 \to$	$3 \to$	$3 \to$	$3 \to 0$
$4 \to$	$4 \to$	$4 \to$	$4 \to$	$4 \to 0$
$5 \to$	$5 \to$	$5 \to$	$5 \to 6$	$5 \to 6$
$6 \to$	$6 \to$	$6 \to$	$6 \to 5$	$6 \to 5$
$7 \to$	$7 \to$	$7 \to 7$	$7 \to 7$	$7 \to 7$
$8 \to$	$8 \to 9$	$8 \to 9$	$8 \to 9$	$8 \to 9$
$9 \to$	$9 \to 10$	$9 \to 10$	$9 \to 10$	$9 \to 10$
$10 \to$	$10 \to 8$	$10 \to 8$	$10 \to 8$	$10 \to 8$

***Theorem* 3** There are exactly 2^N-1 different global orbit patterns in $\mathscr{F}_N$.

That is

$$\sharp\mathscr{G}(\mathscr{F}_N) = 2^N - 1. \tag{10}$$

For example, for $N = 4$, $\sharp\mathscr{G}(\mathscr{F}_4) = 2^4 - 1 = 15$.

Proof Let p an integer between 1 and N. Consider the set $L(p,N)$ of p-tuples $(\alpha_1,\ldots,\alpha_p) \in (\mathbb{N}^*)^p$ such that $\alpha_1 + \ldots + \alpha_p \leq N$.
We write $L(N) = \{L(p,N), p = 1 \ldots N\}$. $L(N)$ and $\mathscr{G}(\mathscr{F}_N)$ have the same elements. Then
$\sharp\mathscr{G}(\mathscr{F}_N) = \sum_{p=1}^{p=N} \sharp L(p,N) = \sum_{p=1}^{p=N} \begin{pmatrix} N \\ p \end{pmatrix} = 2^N - 1.$

Ordering the global orbit patterns

We define an order relation on $\mathscr{G}(\mathscr{F}_N)$.

***Proposition* 1** Let A and B be two global orbit patterns of $\mathscr{G}(\mathscr{F}_N)$.
We define the relation $\prec$ on the set $\mathscr{G}(\mathscr{F}_N)$ by

$$A \prec B \ \textit{iff} \ Tr(A) < Tr(B)$$

Then the set $(\mathscr{G}(\mathscr{F}_N), \prec)$ is totally ordered.

Proof As the order $\prec$ refers to the natural order of $\mathbb{N}$, the proof is obvious.

Let $r \geq 1$, $p \geq 1$ be two integers. Let $[\omega_1,\ldots,\omega_p]$ and $[\omega'_1,\ldots,\omega'_r]$ be two global orbit patterns of $\mathscr{G}(\mathscr{F}_N)$. For example, if $p < r$, in order to compare them, we admit that we can fill $[\omega_1,\ldots,\omega_p]$ with $r - p$ zeros and write $[\omega_1,\ldots,\omega_p] = [\omega_1,\ldots,\omega_p,0,\ldots,0]$.

***Proposition* 2** Let $r \geq 1$, $p \geq 1$ be two integers such that $p \leq r$. Let $A = [\omega_1,\ldots,\omega_p]$ and $B = [\omega'_1,\ldots,\omega'_r]$ be two global orbit patterns.

- If $r = p = 1$ and $\omega_1 < \omega'_1$ then $A \prec B$.
- If $r \geq 2$ then
  - ∗ If $\omega_1 < \omega'_1$ then $A \prec B$.
  - ∗ If $\omega_1 = \omega'_1$ then there exists $K \in [\![2;r]\!]$ such that $\omega_K \neq \omega'_K$ and $\forall i < K$, $\omega_i = \omega'_i$.
    - If $|A| < |B|$, then $A \prec B$.
    - If $|A| = |B|$, then if $\omega_K < \omega'_K$ then $A \prec B$.

For example, for $N = 5$, the global orbit patterns are in increasing order : $[1] \prec [1_{\tilde{2}}] \prec [1_{\tilde{3}}] \prec [1,2] \prec [1_{\tilde{4}}] \prec [1_{\tilde{2}},2] \prec [1,2,1] \prec [1,3] \prec [1_{\tilde{5}}] \prec [1_{\tilde{3}},2] \prec [1_{\tilde{2}},2,1] \prec [1_{\tilde{2}},3] \prec$

$[1,2,1_{\tilde{2}}] \prec [1,2_{\tilde{2}}] \prec [1,3,1] \prec [1,4] \prec [2] \prec [2,1] \prec [2,1_{\tilde{2}}] \prec [2_{\tilde{2}}] \prec [2,1_{\tilde{3}}] \prec [2,1,2] \prec [2_{\tilde{2}},1] \prec [2,3] \prec [3] \prec [3,1] \prec [3,1_{\tilde{2}}] \prec [3,2] \prec [4] \prec [4,1] \prec [5]$.

Algorithm for ordering the global orbit patterns : a pseudo-decimal order

The Table 10 gives a method for ordering the gop : indeed, we consider each gop as if each one represents a decimal number : we begin to order them in considering the first order ω_1. Considering two gops $A = [\omega_1, \ldots, \omega_p]$ and $A' = [\omega'_1, \ldots, \omega'_r]$, if $\omega_1 < \omega'_1$, then $A \prec A'$. For example, $[2,1,2] \prec [4,1]$. If $\omega_1 = \omega'_1$ and $|A| - \omega_1 < |A'| - \omega'_1$, then $A \prec A'$. For example to compare the gop $[1,2]$ and the gop $[1_{\tilde{4}}]$, we say that the first order ω_1 stands for the unit digit - which is $\omega_1 = 1$ here, then the decimal digits are respectively 0.2 and 0.111. We calculate for each of them the modulus-ω_1 : we find $|[1,2]| - 1 = 2$ and $|[1_{\tilde{4}}]| - 1 = 3$, thus $[1,2] \prec [1_{\tilde{4}}]$. Finally, if $\omega_1 = \omega'_1$ and $|A| - \omega_1 = |A'| - \omega'_1$, then also we use the order of the decimal part. For example, $[1_{\tilde{5}}] \prec [1,2,1_{\tilde{2}}$ because $1.1111 < 1.211$. Applying this process, we have the sequence of the ordered gop for $N = 4$ given in the previous paragraph.

TABLE 10. Ordered gop for $N = 5$ with modulus and modulus-ω_1

Gop	Modulus	Modulus-ω_1	Gop	Modulus	Modulus-ω_1
$[1]$	1	0	$[2]$	2	0
$[1_{\tilde{2}}]$	2	1	$[2,1]$	3	1
$[1_{\tilde{3}}]$	3	2	$[2,1_{\tilde{2}}]$	4	2
$[1,2]$	3	2	$[2_{\tilde{2}}]$	4	2
$[1_{\tilde{4}}]$	4	3	$[2,1_{\tilde{3}}]$	5	3
$[1_{\tilde{2}},2]$	4	3	$[2,1,2]$	5	3
$[1,2,1]$	4	3	$[2_{\tilde{2}},1]$	5	3
$[1,3]$	4	3	$[2,3]$	5	3
$[1_{\tilde{5}}]$	5	4			
$[1_{\tilde{3}},2]$	5	4	$[3]$	3	0
$[1_{\tilde{2}},2,1]$	5	4	$[3,1]$	4	1
$[1_{\tilde{2}},3]$	5	4	$[3,1_{\tilde{2}}]$	5	2
$[1,2,1_{\tilde{2}}]$	5	4	$[3,2]$	5	2
$[1,2_{\tilde{2}}]$	5	4			
$[1,3,1]$	5	4	$[4]$	4	0
$[1,4]$	5	4	$[4,1]$	5	1
			$[5]$	5	1

For example, for $N = 5$, we construct one branch of a tree with $\omega_1 = 1$ (see Fig. 3) : each vertex is an ordered orbit, the modulus of the gop is written on the last edge. However, the sequence of ordered gop differs from the natural downward lecture of the tree and has to be done following the algorithm.

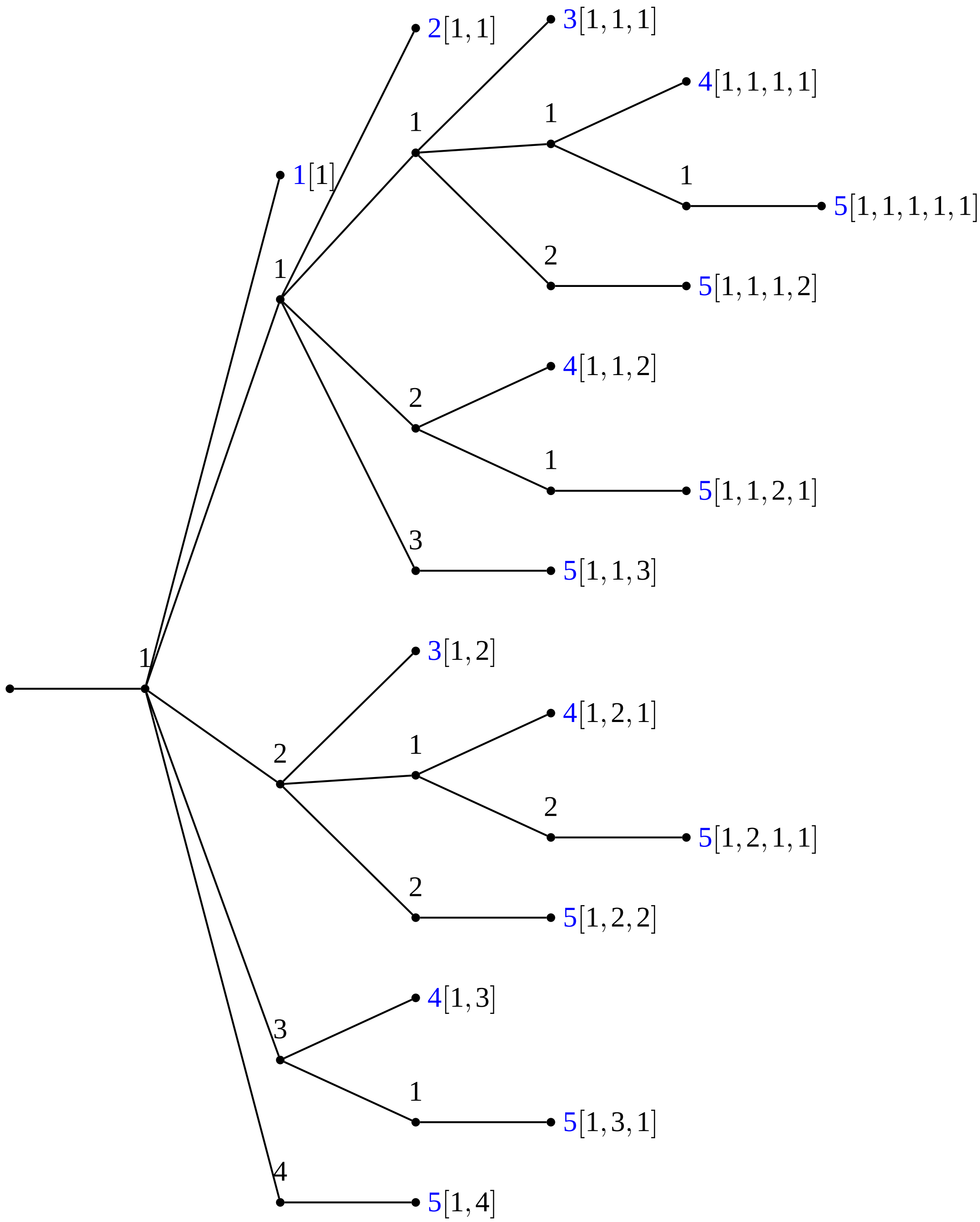

FIGURE 3. Branch of the tree for the construction of the gop with $\omega_1 = 1$ on $\mathscr{G}(\mathscr{F}_5)$

CARDINAL OF CLASSES

In this section we emphasize some closed formulas giving the cardinal of classes of gop. Recalling first the already known formula for the class $[\overline{1_{\widetilde{k}}}]_N$ for which we give a detailed proof, we consider the case were the class possesses exactly one k-cycle, the case with only two cycles belonging to the class and finally the main general formula of any cycles with any length. We give rigorous proof of all. The general formula is very interesting in the sense that even using computer network it is impossible to check every function of $\mathscr{F}_N$ when N is larger than 100.

Discrete maps with 1-cycle only

The theorem 4 gives the number of discrete maps of $\mathscr{F}_N$ which have only fixed points and no cycles of length greater than one. This formula is explicit in [2] and [9]. A complete proof is given here in detail.

***Theorem* 4** Let k be an integer between 1 and N. The number of functions whose global orbit pattern is $[1_{\tilde{k}}]_N$ (i.e. belonging to the class $[\overline{1_{\tilde{k}}}]_N$) is $\begin{pmatrix} N-1 \\ N-k \end{pmatrix} N^{N-k}$.

That is

$$\sharp[\overline{1_{\tilde{k}}}]_N = \begin{pmatrix} N-1 \\ N-k \end{pmatrix} N^{N-k}. \tag{11}$$

Proof Let k be a non-zero integer. Let f be a function of $\mathscr{F}_N$. There are $\begin{pmatrix} N \\ k \end{pmatrix}$ possibilities to choose k fixed points. There remain $N-k$ points. Let p be an integer between 1 and $N-k$. We assume that p points are directly connected to the k fixed points. For each of them, there are k manners to choose one fixed point. There are k^p ways to connect directly p points to k fixed points. There remains $N-k-p$ points that we must connect to the p points. There are $\sharp[1_{\tilde{p}}]_{N-k}$ functions. Finally, the number of functions with k fixed points is $\binom{N}{k} \sum_{p=1}^{N-k} k^p \ \sharp[\overline{1_{\tilde{p}}}]_{N-k}$. We now prove recursively on N for every $0 \leq k \leq N$ that $\sharp[\overline{1_{\tilde{k}}}]_N = \begin{pmatrix} N-1 \\ N-k \end{pmatrix} N^{N-k}$. We have $\sharp\overline{[1]}_1 = 1$. The formula is true.

We suppose that $\forall k \leq N \ \sharp[\overline{1_{\tilde{k}}}]_N = \begin{pmatrix} N-1 \\ N-k \end{pmatrix} N^{N-k}$.

Let X be a set with $N+1$ elements. We look for the functions of $\mathscr{F}_{N+1}$ which have k fixed points. Thanks to the previous reasoning, we have

$\sharp[\overline{1_{\tilde{k}}}]_{N+1} = \begin{pmatrix} N+1 \\ k \end{pmatrix} \sum_{p=1}^{N+1-k} k^p \ \sharp[\overline{1_{\tilde{p}}}]_{N+1-k}$.

$\sharp[\overline{1_{\tilde{k}}}]_{N+1} = \begin{pmatrix} N+1 \\ k \end{pmatrix} \sum_{p=1}^{N+1-k} k^p \ \sharp[\overline{1_{\tilde{p}}}]_{N-(k-1)}$.

We use the recursion assumption.

$\sharp[\overline{1_{\tilde{k}}}]_{N+1} = \binom{N+1}{k} \sum_{p=1}^{N+1-k} k^p \binom{N-k}{p-1} (N-k+1)^{N-k+1-p}$.

$$\sharp[\overline{1_{\widetilde{k}}}]_{N+1} = k\binom{N+1}{k}\sum_{p=0}^{N-k}\binom{N-k}{p}\, k^p (N-k+1)^{N-k-p}.$$
$$\sharp[\overline{1_{\widetilde{k}}}]_{N+1} = k\binom{N+1}{k}(N+1)^{N-k}.$$
$$\sharp[\overline{1_{\widetilde{k}}}]_{N+1} = \binom{N}{k-1}(N+1)^{N-k+1}.$$
$$\sharp[\overline{1_{\widetilde{k}}}]_{N+1} = \binom{N}{N-k+1}(N+1)^{N-k+1}.\ \text{q.e.d.}$$

Discrete maps with k-cycle

We look now for the number of functions with exactly one k-cycle.

***Theorem* 5** Let k be an integer between 1 and N. The number of functions whose global orbit pattern is $[k]_N$ is $\sharp[\overline{1_{\widetilde{k}}}]_N \times (k-1)!$.

i.e.

$$\sharp[\overline{k}]_N = \sharp[\overline{1_{\widetilde{k}}}]_N \times (k-1)!. \tag{12}$$

Proof There are $\binom{N}{k}$ ways of choosing k elements among N. Then, there are $(k-1)!$ choices for the image of those k elements in order to constitute a k-cycle by f. We must now count the number of ways of connecting directly or not the remaining $N-k$ elements to the k-cycle. We established already this number which is equal to $\sum_{p=1}^{N-k} k^p\ \sharp[\overline{1_{\widetilde{p}}}]_{N-k}$. Finally, we have $\sharp[\overline{k}]_N = (k-1)!\binom{N}{k}\sum_{p=1}^{N-k} k^p\ \sharp[\overline{1_{\widetilde{p}}}]_{N-k}$. That is, $\sharp[k]_N = \sharp[\overline{1_{\widetilde{k}}}]_N \times (k-1)!$. q.e.d.

Discrete maps with only two cycles

We give the number of functions with only two cycles.

***Theorem* 6** Let $N \geq 2$. Let p and q be two non-zero integers such that $p+q \leq N$. Then,

$$\sharp[\overline{p,q}]_N = \sharp[\overline{1_{\widetilde{p+q}}}]_N \frac{(p+q-1)!}{q} = \frac{(N-1)!\ N^{N-(p+q)}}{(N-(p+q))!\ q}. \tag{13}$$

Proof We consider a function f which belongs to the class $[\overline{1_{\widetilde{p+q}}}]_N$. We search the number of functions constructed from f whose gop is $[p,q]_N$. From the p fixed points of

f, we construct a p-cycle. Thus, there are $\binom{p+q-1}{p-1}$ ways to choose $p-1$ integers among the $p+q-1$ fixed points. Counting the first given fixed point of f, we have p points which allow to construct $(p-1)!$ functions with a p-cycle. Then there remain q points which give $(q-1)!$ different functions with a q-cycle. Finally, the number of functions whose gop is $[p,q]_N$ is : $\binom{p+q-1}{p-1}(p-1)!(q-1)!$ that is the formula $\frac{(p+q-1)!}{q}$.

Remark We notice that for all k non-zero integer such that $k \leq N-1$, $\sharp\overline{[k,1]}_N = \sharp\overline{[k+1]}_N$.

General case : discrete maps with cycles of any length

We introduce now the main theorem of the section which gives the number of gop of discrete maps thanks to a closed formula.

Given a global orbit pattern α, the next theorem gives a formula which gives the number of functions which belong to $\overline{\alpha}$.

Theorem 7 Let $p \geq 2$ be an integer. Let $[\omega_1, \ldots, \omega_p]_N$ be a gop of $\mathscr{G}(\mathscr{F}_N)$. Then,

$$\sharp\overline{[\omega_1,\ldots,\omega_p]}_N = \sharp\overline{[1_{\widetilde{\omega_1+\ldots+\omega_p}}]}_N \frac{(\omega_1+\ldots+\omega_p-1)!}{\omega_p \times (\omega_{p-1}+\omega_p) \times \ldots \times (\omega_2+\ldots+\omega_p)} \tag{14}$$

$$\sharp\overline{[\omega_1,\ldots,\omega_p]}_N = \frac{(N-1)!\ N^{N-(\omega_1+\ldots+\omega_p)}}{(N-(\omega_1+\ldots+\omega_p))!\ \prod\limits_{k=2}^{p}(\sum\limits_{j=k}^{p}\omega_j)} \tag{15}$$

Proof We consider a function f which belongs to $\overline{[1_{\widetilde{\omega_1+\ldots+\omega_p}}]}_N$. We search the number of functions constructed from f whose gop is $[\omega_1,\ldots,\omega_p]_N$. From the ω_1 fixed points of f, we construct a ω_1-cycle. Thus, there are $\binom{\omega_1+\ldots+\omega_p-1}{\omega_1-1}$ ways to choose ω_1-1 integers among the $\omega_1+\ldots+\omega_p-1$ fixed points. Counting the first given fixed point of f, we have ω_1 points which allow to construct $(\omega_1-1)!$ functions with a ω_1-cycle. Then, the first fixed point of f which has not be chosen for the ω_1-cycle, will belong to the ω_2-cycle. Thus, there are $\binom{\omega_2+\ldots+\omega_p-1}{\omega_2-1}$ ways to choose ω_2-1 integers among the $\omega_2+\ldots+\omega_p-1$ fixed points. So we have ω_2 points which allow to construct $(\omega_2-1)!$ functions with a ω_2-cycle. We keep going on that way until there remain ω_p fixed points which allow to construct $(\omega_p-1)!$ functions with a ω_p-cycle. Finally, we have constructed :
$\binom{\omega_1+\ldots+\omega_p-1}{\omega_1-1}(\omega_1-1)!\binom{\omega_2+\ldots+\omega_p-1}{\omega_2-1}(\omega_2-1)! \times \ldots \times \binom{\omega_{p-1}+\omega_p-1}{\omega_{p-1}-1}(\omega_{p-1}-1)!(\omega_p-1)!$
functions. We simplify and obtain the formula.

***Corollary* 1** Let p be a non-zero integer. Let $[\omega_1,\ldots,\omega_p]_N$ be a gop of $\mathscr{G}(\mathscr{F}_N)$. We suppose that there exists j such that $\omega_j \geq 2$. Let h be an integer between 1 and $\omega_j - 1$. Then

$$\sharp\overline{[\omega_1,\ldots,\omega_j,\ldots,\omega_p]}_N = \sharp\overline{[\omega_1,\ldots,\omega_j-h,h,\omega_{j+1},\ldots,\omega_p]}_N \times (h+\omega_{j+1}+\ldots+\omega_p). \tag{16}$$

Proof $\sharp\overline{[\omega_1,\ldots,\omega_j-h,h,\omega_{j+1},\ldots,\omega_p]}_N \times (h+\omega_{j+1}+\ldots+\omega_p) = \sharp\overline{[1_{\widetilde{\omega_1+\ldots+\omega_p}}]}_N$
$\times \frac{(\omega_1+\ldots+\omega_p-1)!(h+\omega_{j+1}+\ldots+\omega_p)}{\omega_p(\omega_{p-1}+\omega_p)\ldots(\omega_{j+1}+\ldots+\omega_p)(h+\omega_{j+1}+\ldots+\omega_p)(\omega_j+\omega_{j+1}+\ldots+\omega_p)\times\ldots\times(\omega_2+\ldots+\omega_p)}$.

We simplify and we exactly obtain

$$\sharp\overline{[\omega_1,\ldots,\omega_j-h,h,\omega_{j+1},\ldots,\omega_p]}_N \times (h+\omega_{j+1}+\ldots+\omega_p) = \sharp\overline{[\omega_1,\ldots,\omega_j,\ldots,\omega_p]}_N.$$

Examples :

$\sharp\overline{[2_{\tilde{2}},1,3]}_{11} = 11,180,400.$

$\sharp\overline{[5,2,10,8,15,2,3]}_{50} = 29,775,702,147,667,389,218,762,343,520,975,006,$ $348,329,578,044,480,000,000,000,000,000.$

$\sharp\overline{[5,2,10,8,15,2,3]}_{50} \cong 2.98\times10^{63}$ among the 8.88×10^{84} functions of $\mathscr{F}_{50}$.

FUNCTIONS WITH LOCAL PROPERTIES

Locally rigid functions

Obviously it is not possible to transpose to the functions on finite sets the notions of continuity and derivability which play a dramatic role in mathematical analysis since several centuries. In fact the class $\mathscr{C}_0(I)$ of the continuous functions on the real interval I is a very small subset of the set $I^{\mathbb{R}}$ of all the functions on I. Hence by analogy to this fact and trying to mimic some others properties of continuous functions, we introduce some subsets of particular functions of $\mathscr{F}_N$, which have local properties such as locally bounded range in a sense we precise further. Limiting the range of the function in a neighbourhood of any point of the interval induces a kind of "rigidity" of the function, hence we call these functions locally rigid functions. In these subsets, the gop are found to be fully efficient in order to describe very precisely the dynamics of the orbits. We first consider the very simple subset $\mathscr{L}\mathscr{R}_{1,N}$ of functions for which the difference between $f(p)$ and $f(p+1)$ is drastically bounded. In next subsection we consider more sophisticated subsets.

We consider the set :

$\mathscr{LR}_{1,N}$=$\{f\in\mathscr{F}_N$ such that $\forall p, 0 \leq p \leq N-2, |f(p)-f(p+1)| \leq 1\}$.

Orbits of $\mathscr{LR}_{1,N}$

***Theorem* 8** If $f \in \mathscr{LR}_{1,N}$ then f has only periodic orbits of order 1 or 2.

Proof We suppose that $f \in \mathscr{LR}_{1,N}$ has a 3-cycle. We denote $(a; f(a); f^2(a))$ taking a the smallest value of the 3-cycle. If $a < f(a) < f^2(a)$ then there exist two non-zero integers e and e' such that $f(a) = a+e$ and $f^2(a) = f(a)+e'$. Thus, $f^2(a)-e' \leq f^3(a) \leq f^2(a)+e'$. That is $f(a) \leq a \leq f(a)+2e'$. And finally we have the relation $a+e \leq a$ which is impossible.
If $a < f^2(a) < f(a)$ then there exist two non-zero integers e and e' such that $f^2(a) = a+e$ and $f(a) = f^2(a)+e'$. Thus, $f(a)-e \leq f^3(a) \leq f(a)+e$. That is $f(a)-e \leq a \leq f(a)+e$. But $f(a)-e = a+e'$. And finally we have the relation $a+e' \leq a$ which is impossible.
We can prove in the same way that the function f can't have either 3-cycle or greater order cycle than 3.

Numerical results and conjectures

We have done numerical studies of the $\mathscr{G}(\mathscr{LR}_{1,N})$ for $N = 1$ to 16, using the brute force of a desktop computer (i.e. checking every function belonging to these sets).

The Tables 11, 12, 13, 14, 15 and 16 show the sequences for $\mathscr{LR}_{1,1}$ to $\mathscr{LR}_{1,16}$.

In theses Tables we display in the first column all the gop of $\mathscr{G}(\mathscr{LR}_{1,N})$ for every value of N. For a given N, there are two columns; the left one displays the cardinal of every existing class of gop (- stands for non existing gop). Instead the second shows more regularity, displaying on the row of the gop $[2_{\tilde{k}}]$ the sum of the cardinals of all the classes of the gop of the form $\underbrace{[2,2,\ldots\ldots,\underbrace{1}_{i\text{th}},\ldots,2]}_{k+1 \text{ orders}}$ which exist.

Then we are able to formulate some statements which have not yet been proved.

***Statement* 1**

$$\sharp[1_{\tilde{k}}]_{\mathscr{LR}_{1,N}} = \sharp[1_{\widetilde{k+1}}]_{\mathscr{LR}_{1,N+1}} \text{ for } k \leq \frac{N+1}{2}. \tag{17}$$

TABLE 11. Numbering the locally rigid functions for $f \in \mathscr{L}\mathscr{R}_{1,1}$, $f \in \mathscr{L}\mathscr{R}_{1,2}$, $f \in \mathscr{L}\mathscr{R}_{1,3}$, $f \in \mathscr{L}\mathscr{R}_{1,4}$.

g.o.p.	N=1	N=1	N=2	N=2	N=3	N=3	N=4	N=4
Total number		1		4		17		68
$[1]$	1	+	2	+	7	+	26	+
$[1_{\tilde{2}}]$	-	+	1	+	4	+	14	+
$[1_{\tilde{3}}]$	-	+	-	+	1	+	4	+
$[1_{\tilde{4}}]$	-	+	-	+	-	+	1	+
$[2]$	-	+	1	1	4	4	18	18
$[2,1]$	-	+	-	+	1	1	3	4
$[1,2]$	-	+	-	+	-	+	1	+
$[2_{\tilde{2}}]$	-	+	-	+	-	+	1	1

TABLE 12. Numbering the locally rigid functions for $f \in \mathscr{L}\mathscr{R}_{1,5}$, $f \in \mathscr{L}\mathscr{R}_{1,6}$, $f \in \mathscr{L}\mathscr{R}_{1,7}$.

g.o.p.	N=5	N=5	N=6	N=6	N=7	N=7
Total number		259		950		387
$[1]$	95	+	340	+	1,193	+
$[1_{\tilde{2}}]$	50	+	174	+	600	+
$[1_{\tilde{3}}]$	16	+	58	+	204	+
$[1_{\tilde{4}}]$	4	+	16	+	60	+
$[1_{\tilde{5}}]$	1	+	4	+	16	+
$[1_{\tilde{6}}]$	-	+	1	+	4	+
$[1_{\tilde{7}}]$	-	+	-	+	1	+
$[2]$	70	70	264	264	952	952
$[2,1]$	12	18	45	70	166	264
$[1,2]$	6	+	25	+	98	+
$[2_{\tilde{2}}]$	4	4	18	18	70	70
$[2_{\tilde{2}},1]$	1	1	4	4	17	18
$[1,2_{\tilde{2}}]$	-	+	-	+	1	+
$[2,1,2]$	-	+	-	+	-	+
$[2_{\tilde{3}}]$	-	+	1	1	4	4
$[2_{\tilde{3}},1]$	-	+	-	+	1	1

Statement **2**

$$\sharp[\overline{2_{\tilde{k}}}]_{\mathscr{L}\mathscr{R}_{1,N}} = \sharp[\overline{2_{\widetilde{k+1}}}]_{\mathscr{L}\mathscr{R}_{1,N+2}} \text{ for } k \leq \frac{N}{2}. \tag{18}$$

Statement **3**

$$\sharp[\overline{2_{\tilde{k}}}]_{\mathscr{L}\mathscr{R}_{1,N}} = \sharp[\overline{2_{\tilde{k}},1}]_{\mathscr{L}\mathscr{R}_{1,N+1}} \text{ for } 2k \leq N \leq 3k-1. \tag{19}$$

TABLE 13. Numbering the locally rigid functions for $f \in \mathscr{L}\mathscr{R}_{1,8}$, $f \in \mathscr{L}\mathscr{R}_{1,9}$, $f \in \mathscr{L}\mathscr{R}_{1,10}$.

g.o.p.	N=8	N=8	N=9	N=9	N=10	N=10
Total number		11,814		40,503		13,6946
$[1]$	4,116	+	14,001	+	47,064	+
$[1_{\widetilde{2}}]$	2,038	+	6,852	+	22,806	+
$[1_{\widetilde{3}}]$	700	+	2,366	+	7,896	+
$[1_{\widetilde{4}}]$	214	+	742	+	2,520	+
$[1_{\widetilde{5}}]$	60	+	216	+	754	+
$[1_{\widetilde{6}}]$	16	+	60	+	216	+
$[1_{\widetilde{7}}]$	4	+	16	+	60	+
$[1_{\widetilde{8}}]$	1	+	4	+	16	+
$[1_{\widetilde{9}}]$	-	+	1	+	4	+
$[1_{\widetilde{10}}]$	-	+	-	+	1	+
$[2]$	3,356	3,356	11,580	11,580	39,364	39,364
$[2,1]$	590	952	2,062	3,356	7,072	11,580
$[1,2]$	362	+	1,294	+	4,508	+
$[2_{\widetilde{2}}]$	264	264	952	952	3,356	3,356
$[2_{\widetilde{2}},1]$	62	70	222	264	770	952
$[1,2_{\widetilde{2}}]$	6	+	28	+	113	+
$[2,1,2]$	2	+	14	+	69	+
$[2_{\widetilde{3}}]$	18	18	70	70	264	264
$[2_{\widetilde{3}},1]$	4	4	18	18	69	70
$[1,2_{\widetilde{3}}]$	-	+	-	+	1	+
$[2,1,2_{\widetilde{2}}]$	-	+	-	+	-	+
$[2_{\widetilde{2}},1,2]$	-	+	-	+	-	+
$[2_{\widetilde{4}}]$	1	1	4	4	18	18
$[2_{\widetilde{4}},1]$	-	+	1	1	4	4
$[1,2_{\widetilde{4}}]$	-	+	-	+	-	+
$[2,1,2_{\widetilde{3}}]$	-	+	-	+	-	+
$[2_{\widetilde{2}},1,2_{\widetilde{2}}]$	-	+	-	+	-	+
$[2_{\widetilde{3}},1,2]$	-	+	-	+	-	+
$[2_{\widetilde{5}}]$	-	+	-	+	1	1

***Statement* 4**

$$
\begin{aligned}
\sharp[\overline{2_{\widetilde{k}}}]_{\mathscr{L}\mathscr{R}_{1,N}} &= \sum_{i=1}^{k+1} \sharp[\underbrace{\overline{2,2,\ldots\ldots,\underbrace{1}_{i\text{th}},\ldots,2}}_{k+1 \text{ orders}}]_{\mathscr{L}\mathscr{R}_{1,N}} \text{ for } 2k+1 \leq N \\
&= \sum_{i=1}^{k+1} \sharp[\overline{2_{\widetilde{i-1}},1,2_{\widetilde{k-i+1}}}]_{\mathscr{L}\mathscr{R}_{1,N}} \text{ for } 2k+1 \leq N \qquad (20)
\end{aligned}
$$

TABLE 14. Numbering the locally rigid functions for $f \in \mathscr{L}\mathscr{R}_{1,11}$, $f \in \mathscr{L}\mathscr{R}_{1,12}$, $f \in \mathscr{L}\mathscr{R}_{1,13}$.

g.o.p.	N=11	N=11	N=12	N=12	N=13	N=13
Total number		457,795		1,515,926		4,979,777
$[1]$	156,629	+	516,844	+	1,693,073	+
$[1_{\widetilde{2}}]$	75,292	+	246,762	+	803,706	+
$[1_{\widetilde{3}}]$	26,098	+	85,556	+	278,580	+
$[1_{\widetilde{4}}]$	8,434	+	27,904	+	91,488	+
$[1_{\widetilde{5}}]$	2,756	+	8,658	+	28,738	+
$[1_{\widetilde{6}}]$	756	+	2,590	+	8,730	+
$[1_{\widetilde{7}}]$	216	+	756	+	2,592	+
$[1_{\widetilde{8}}]$	60	+	216	+	756	+
$[1_{\widetilde{9}}]$	16	+	60	+	216	+
$[1_{\widetilde{10}}]$	4	+	16	+	60	+
$[1_{\widetilde{11}}]$	1	+	4	+	16	+
$[1_{\widetilde{12}}]$	-	+	1	+	4	+
$[1_{\widetilde{13}}]$	-	+	-	+	1	+
$[2]$	132,104	132,104	438,846	438,846	1,445,258	1,445,258
$[2,1]$	23,941	39,364	80,108	132,104	265,548	438,846
$[1,2]$	15,423	+	51,996	+	173,298	+
$[2_{\widetilde{2}}]$	11,580	11,580	39,364	39,364	132,104	132,104
$[2_{\widetilde{2}},1]$	2,634	3,356	8,883	11,580	29,659	39,364
$[1,2_{\widetilde{2}}]$	429	+	1,555	+	5,478	+
$[2,1,2]$	293	+	1,142	+	4,227	+
$[2_{\widetilde{3}}]$	952	952	3,356	3,356	11,580	11,580
$[2_{\widetilde{3}},1]$	255	264	899	952	3,098	3,356
$[1,2_{\widetilde{3}}]$	7	+	35	+	152	+
$[2,1,2_{\widetilde{2}}]$	2	+	16	+	86	+
$[2_{\widetilde{2}},1,2]$	-	+	2	+	20	+
$[2_{\widetilde{4}}]$	70	70	264	264	952	952
$[2_{\widetilde{4}},1]$	18	18	70	70	263	264
$[1,2_{\widetilde{4}}]$	-	+	-	+	1	+
$[2,1,2_{\widetilde{3}}]$	-	+	-	+	-	+
$[2_{\widetilde{2}},1,2_{\widetilde{2}}]$	-	+	-	+	-	+
$[2_{\widetilde{3}},1,2]$	-	+	-	+	-	+
$[2_{\widetilde{5}}]$	4	4	18	18	70	70
$[2_{\widetilde{5}},1]$	1	1	4	4	18	18
$[1,2_{\widetilde{5}}]$	-	+	-	+	-	+
$[2_{\widetilde{6}}]$	-	+	1	1	4	4
$[2_{\widetilde{6}},1]$	-	+	-	+	1	1

***Statement* 5**

$$\sharp[\overline{1_{\widetilde{N-k+1}}}]_{\mathscr{L}\mathscr{R}_{1,N}} = \begin{cases} 1 & \text{if } k=1 \\ 2 & \text{if } k=2 \\ \left(\frac{4}{27}\right)(k+1)\times 3^k & \text{for } 3 \le k \le \frac{N+1}{2} \end{cases} \tag{21}$$

TABLE 15. Numbering the locally rigid functions for $f \in \mathscr{LR}_{1,14}$, $f \in \mathscr{LR}_{1,15}$.

g.o.p.	N=14	N=14	N=15	N=15
Total number		16,246,924		52,694,573
$[1]$	5,511,218	+	17,841,247	+
$[1_{\tilde{2}}]$	2,603,258	+	8,391,360	+
$[1_{\tilde{3}}]$	901,802	+	2,904,592	+
$[1_{\tilde{4}}]$	297,728	+	962,888	+
$[1_{\tilde{5}}]$	94,440	+	307,848	+
$[1_{\tilde{6}}]$	29,050	+	95,676	+
$[1_{\tilde{7}}]$	8,746	+	29,140	+
$[1_{\tilde{8}}]$	2,592	+	8,748	+
$[1_{\tilde{9}}]$	756	+	2,592	+
$[1_{\widetilde{10}}]$	216	+	756	+
$[1_{\widetilde{11}}]$	60	+	216	+
$[1_{\widetilde{12}}]$	16	+	60	+
$[1_{\widetilde{13}}]$	4	+	16	+
$[1_{\widetilde{14}}]$	1	+	4	+
$[1_{\widetilde{15}}]$	-	+	1	+
$[2]$	4,725,220	4,725,220	15,352,392	15,352,392
$[2,1]$	873,149	1,445,258	2,851,350	+
$[1,2]$	572,109	+	1,873,870	+
$[2_{\tilde{2}}]$	438,846	438,846	1,445,258	1,445,258
$[2_{\tilde{2}},1]$	98,135	132,104	322,310	438,846
$[1,2_{\tilde{2}}]$	18,873	+	63,967	+
$[2,1,2]$	15,096	+	52,569	+
$[2_{\tilde{3}}]$	39,364	39,364	132,104	132,104
$[2_{\tilde{3}},1]$	10,460	11,580	34,845	39,364
$[1,2_{\tilde{3}}]$	605	+	2,282	+
$[2,1,2_{\tilde{2}}]$	389	+	1,596	+
$[2_{\tilde{2}},1,2]$	126	+	641	+
$[2_{\tilde{4}}]$	3,356	3,356	11,580	11,580
$[2_{\tilde{4}},1]$	942	952	3,292	3,356
$[1,2_{\tilde{4}}]$	8	+	44	+
$[2,1,2_{\tilde{3}}]$	2	+	18	+
$[2_{\tilde{2}},1,2_{\tilde{2}}]$	-	+	2	+
$[2_{\tilde{3}},1,2]$	-	+	-	+
$[2_{\tilde{5}}]$	264	264	952	952
$[2_{\tilde{5}},1]$	70	70	264	264
$[1,2_{\tilde{5}}]$	-	+	-	+
$[2_{\tilde{6}}]$	18	18	70	70
$[2_{\tilde{6}},1]$	4	4	18	18
$[2_{\tilde{7}}]$	1	1	4	4
$[2_{\tilde{7}},1]$	-	+	1	1

Remark We call $u_k = \sharp[\overline{1_{\widetilde{N-k+1}}}]_{\mathscr{LR}_{1,N}}$. For $k > 2$, then u_k is the sequence A120926 On-line Encyclopedia of integer Sequences : it is the number of sequences where 0 is isolated in ternary words of length N written with $\{0,1,2\}$.

These statements show that first the set $\mathscr{LR}_{1,N}$ is an interesting set to be considered for dynamical systems and secondly the gop are fruitful in this study. However the set

$$\mathscr{LR}_{2,N}=\{f\in\mathscr{F}_N \text{ such that } \forall p, 0\leq p\leq N-2, |f(p)-f(p+1)|\leq 2\}$$

is too much large to give comparable results. Then we introduce more sophisticated sets we call sets with locally bounded range which more or less correspond to an analogue of the discrete convolution product of the local variation of f with a compact support function $\overrightarrow{\alpha_t}$.

Orbits and patterns of locally rigid function sets

Consider now the set :

$\mathscr{LR}_{\overrightarrow{\alpha_t},q,N}=\{f\in\mathscr{F}_N$ such that $\forall p, 0\leq p\leq N-r-1, \sum_{r=1}^{r=t}\alpha_r|f(p)-f(p+r)|\leq q\}\bigcap\{f\in\mathscr{F}_N$ such that $\forall p, t\leq p\leq N-1, \sum_{r=1}^{r=t}\alpha_r|f(p)-f(p-r)|\leq q\}$ for the vector $\overrightarrow{\alpha_t}=(\alpha_1,\alpha_2,\ldots,\alpha_t)\in\mathbb{N}^t$, for $q\in\mathbb{N}$.

TABLE 17. Numerical study of the set $\mathscr{LR}_{\vec{\alpha_t},q,N}$ for $N=10$, $t=5$, $\alpha_1=20$, $\alpha_2=9$, $\alpha_3=5$, $\alpha_4=2$ and $\alpha_5=1$, for $q=20,\ldots,142$

q	maximal period	modulus	gop number	functions number
20	1	1	1	10
26	2	2	3	82
44	2	3	6	21,764
49	3	3	7	48,112
50	3	3	7	53,210
56	3	4	9	208,692
59	4	4	15	330,800
63	4	5	19	626,890
66	4	10	37	952,228
67	4	10	46	1,064,316
72	5	10	50	1,630,018
74	6	10	60	1,816,826
76	6	10	61	2,152,450
77	6	10	88	2,416,368
78	6	10	91	2,762,434
79	6	10	97	3,188,080
80	6	10	99	3,735,666
84	6	10	100	5,876,324
85	6	10	103	6,473,288
87	6	10	105	7,851,728
88	7	10	121	8,644,178
89	8	10	129	9,521,920
91	8	10	136	11,414,556
92	8	10	165	12,454,440
94	8	10	175	14,756,058

Following next page

TABLE 17. (Next)

q	maximal period	modulus	gop number	functions number
95	8	10	177	16,077,780
96	8	10	184	17,208,654
97	8	10	185	18,369,854
98	8	10	188	19,585,746
100	8	10	192	22,083,852
101	8	10	199	23,584,452
102	8	10	204	25,513,892
103	8	10	244	27,912,772
104	8	10	304	30,560,238
105	9	10	333	33,516,466
106	9	10	380	36,682,960
107	9	10	424	40,004,280
108	10	10	491	43,685,352
109	10	10	517	47,655,856
110	10	10	529	51,785,410
111	10	10	562	55,907,120
112	10	10	583	60,341,276
113	10	10	612	64,930,790
114	10	10	647	69,766,178
115	10	10	706	74,989,752
116	10	10	747	80,087,120
117	10	10	791	85,570,272
118	10	10	820	91,206,218
119	10	10	836	97,040,288
120	10	10	852	103,121,916
121	10	10	872	109,650,464
122	10	10	896	116,345,296
123	10	10	919	123,241,156
124	10	10	924	130,360,938
125	10	10	928	137,636,628
126	10	10	930	145,536,068
127	10	10	932	154,370,862
128	10	10	938	164,145,928
129	10	10	960	174,942,026
130	10	10	986	186,438,038
131	10	10	1,006	198,594,118
132	10	10	1,013	211,550,402
133	10	10	1,015	225,324,700
134	10	10	1,021	239,976,118
135	10	10	1,022	255,106,866
137	10	10	1,023	286,726,234
142	10	10	1,023	374,355,356

The functions belonging to these sets show a kind of "rigidity": the less is q, the more "rigid" is the function, this "rigidity" being modulated by the vector $\overrightarrow{\alpha_t}$. Furthermore, the maximal length of a periodic orbit increases with q, and so the number of gop $\sharp\mathscr{G}(\mathscr{L}\mathscr{R}_{\overrightarrow{\alpha_t},q,N})$ and the maximal modulus of the gop.

Remark Using this generalized notation, one has : $\mathscr{LR}_{1,n} = \mathscr{LR}_{1,1,n}$ and $\mathscr{LR}_{2,n} = \mathscr{LR}_{1,2,n}$.

As an example, we explore numerically the case : $N = 10$, $t = 5$, $\alpha_1 = 20$, $\alpha_2 = 9$, $\alpha_3 = 5$, $\alpha_4 = 2$ and $\alpha_5 = 1$, for $q = 20, \dots, 142$. The results are displayed in Table 17. In this Table "modulus" means the maximal modulus of the gop belonging to this set for the corresponding value of q in the row, "gop number" stands for $\sharp\mathscr{G}(\mathscr{LR}_{\overrightarrow{\alpha_t},q,N})$ and "functions number" for $\sharp\mathscr{LR}_{\overrightarrow{\alpha_t},q,N}$. One can point out that for the particular function $\overrightarrow{\alpha_t}$ of the example; it is possible to find 10 intervals $I_1, I_2, \dots, I_{10} \subset \mathbb{N}$ such that if $q \in I_r$ then there is no periodic orbit whose period is strictly greater than r, (e.g., $I_6 = [\![74, 87]\!]$). Furthermore it is possible to split these intervals into subintervals $I_{r,s}$ in which $\sharp\mathscr{G}\left(\mathscr{LR}_{\overrightarrow{\alpha_t},q,N}\right)$ is constant when q thumbs $I_{r,s}$. This is not the case for $\sharp\mathscr{LR}_{\overrightarrow{\alpha_t},q,N}$.

CONCLUSION

A discrete dynamical system associated to a function on finite ordered set X can only exhibit periodic orbits. However the number of the periods and the length of each are not easily predictable. We formalise such a gop as the ordered set of periods when the initial value thumbs X in the increasing order. We can predict by means of closed formulas, the number of gop of the set of all the function from X to itself. We also explore, using the brute force of computers, some subsets of locally rigid functions on X, for which interesting patterns of periodic orbits are found. Further study is needed to understand the behaviour of dynamical systems associated to functions belonging to these sets.

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TABLE 16. Numbering the locally rigid functions for $f \in \mathscr{L}\mathscr{R}_{1,16}$.

g.o.p.	N=16	N=16
Total number		170,028,792
$[1]$	57,477,542	+
$[1_{\widetilde{2}}]$	26,932,398	+
$[1_{\widetilde{3}}]$	9,314,088	+
$[1_{\widetilde{4}}]$	3,097,650	+
$[1_{\widetilde{5}}]$	996,764	+
$[1_{\widetilde{6}}]$	312,456	+
$[1_{\widetilde{7}}]$	96,096	+
$[1_{\widetilde{8}}]$	29,158	+
$[1_{\widetilde{9}}]$	8,748	+
$[1_{\widetilde{10}}]$	2,592	+
$[1_{\widetilde{11}}]$	756	+
$[1_{\widetilde{12}}]$	216	+
$[1_{\widetilde{13}}]$	60	+
$[1_{\widetilde{14}}]$	16	+
$[1_{\widetilde{15}}]$	4	+
$[1_{\widetilde{16}}]$	1	+
$[2]$	49,610,818	49,610,818
$[2,1]$	9,255,822	15,352,392
$[1,2]$	6,096,570	+
$[2_{\widetilde{2}}]$	4,725,220	4,725,220
$[2_{\widetilde{2}},1]$	1,051,686	1,445,258
$[1,2_{\widetilde{2}}]$	213,975	+
$[2,1,2]$	179,597	+
$[2_{\widetilde{3}}]$	438,846	438,846
$[2_{\widetilde{3}},1]$	114,798	132,104
$[1,2_{\widetilde{3}}]$	8,284	+
$[2,1,2_{\widetilde{2}}]$	6,146	+
$[2_{\widetilde{2}},1,2]$	2,876	+
$[2_{\widetilde{4}}]$	39,364	39,364
$[2_{\widetilde{4}},1]$	11,246	11,580
$[1,2_{\widetilde{4}}]$	204	+
$[2,1,2_{\widetilde{3}}]$	106	+
$[2_{\widetilde{2}},1,2_{\widetilde{2}}]$	22	+
$[2_{\widetilde{3}},1,2]$	2	+
$[2_{\widetilde{5}}]$	3,356	3,356
$[2_{\widetilde{5}},1]$	951	952
$[1,2_{\widetilde{5}}]$	1	+
$[2_{\widetilde{6}}]$	264	264
$[2_{\widetilde{6}},1]$	70	70
$[2_{\widetilde{7}}]$	18	18
$[2_{\widetilde{7}},1]$	4	4
$[2_{\widetilde{8}}]$	1	1